MORPHOLOGICAL WAVELETS AND THE COMPLEXITY OF DYADIC TREES

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ABSTRACT

In this paper we reveal a connection between the coefficients of the morphological wavelet transform and complexity measures of dyadic tree representations of level sets. This leads to better understanding of the edge preserving property that has been discovered in both areas. As an immediate application, we examine a depth-adaptive soft thresholding scheme on morphological wavelet coefficients in which the threshold decays geometrically as the resolution increases. A greater decay rate gives greater preference towards unbalanced trees and this can control edge enhancement in denoised signals.

Index Terms— Wavelet transforms, morphological operations, image edge analysis, image enhancement.

1. INTRODUCTION

The morphological wavelet transform ([1],[2]) is a *nonlinear* transform using $\max$ and $\min$ algebraic operations. These operations correspond to morphological operations such as erosion and dilation in images. In [1] and [2] it is established that compared to linear wavelet transforms, morphological wavelet decompositions better preserve geometric structures (such as edges) of an original image in lower resolution approximations. Therefore thresholding the morphological wavelet coefficients can better preserve these features while also removing noise.

Preserving detailed edge information is also very important in level set estimation. In [3], a novel minimax optimal estimator is proposed using regularization on the complexity of a dyadic tree. It is pointed out that an unbalanced tree structure is well suited to capture the edges of level set indicator functions common in actual data. Such a tree can be deep around edges, to capture the edges well, and shallow elsewhere.

In this paper, we draw a connection between these two seemingly different areas. For both clarity and brevity, we restrict attention to scalar signals. In this setting, we prove that the weighted L1 norm of the morphological Haar wavelet coefficients of a function is an integral of certain complexity measures of the function's level sets across thresholds. This leads to some useful insights on both of the above areas, especially on why shrinking morphological wavelet coefficients and level set regularization both result in good edge preservation. As a direct application of this theoretical result, we examine a soft thresholding scheme on the morphological wavelet coefficients in which the threshold decays geometrically as the resolution increases. The decay rate controls the strength of edge detection and enhancement. Our contribution is primarily theoretical. However, we provide some simple denoising experiments to illustrate depth-adaptive thresholding on morphological wavelet coefficients and its relationship to detecting and preserving edges in a signal. The theory developed in this paper offers useful guidance on how to properly choose the decay rate of thresholds.

The paper is organized as follows: We introduce the preliminaries of aforementioned two areas in §2 and §3. A theorem connecting them is established in §4, with immediate illustrations given in §5. We conclude in §6.

2. MORPHOLOGICAL WAVELETS

A discrete wavelet transform recursively decomposes a higher resolution signal into: (a) an approximation signal of lower resolution, and (b) a sequence of detailed coefficients (also of lower resolution). For example, in the Haar wavelet transform, signal $\{\Psi_{k+1}(l)\}_{l=0}^{2^{k+1}-1}$ is decomposed into: (a) $\{\Psi_k(l)\}_{l=0}^{2^k-1}$, and (b) $\{W_{k,l}\}_{l=0}^{2^k-1}$ as follows:

$$\Psi_k(l) = \frac{\Psi_{k+1}(2l) + \Psi_{k+1}(2l+1)}{2}, 0 \le l \le 2^k - 1, \quad (1)$$

$$W_{k,l} = \Psi_{k+1}(2l) - \Psi_{k+1}(2l+1), 0 \le l \le 2^k - 1. \quad (2)$$

The morphological Haar wavelet transform works on the same principle except the averaging operator in (1) is replaced by either a nonlinear $\max$ (denoted $\vee$) or $\min$ (denoted $\wedge$) operator. The $\max$ version of this transform is:

$$\Psi_k^{\vee}(l) = \Psi_{k+1}^{\vee}(2l) \vee \Psi_{k+1}^{\vee}(2l+1), 0 \le l \le 2^k - 1, \quad (3)$$

$$W_{k,l}^{\vee} = \Psi_{k+1}^{\vee}(2l) - \Psi_{k+1}^{\vee}(2l+1), 0 \le l \le 2^k - 1, \quad (4)$$

and the $\min$ version is:

$$\Psi_k^{\wedge}(l) = \Psi_{k+1}^{\wedge}(2l) \wedge \Psi_{k+1}^{\wedge}(2l+1), 0 \le l \le 2^k - 1, \quad (5)$$

$$W_{k,l}^{\wedge} = \Psi_{k+1}^{\wedge}(2l) - \Psi_{k+1}^{\wedge}(2l+1), 0 \le l \le 2^k - 1. \quad (6)$$

As discussed in [1] and [2] the edge information of Ψ_{k+1} is mostly preserved in Ψ_k, therefore controlling the complexity of the signal representation by shrinking $W_{k,l}$ will not compromise the geometric details of the original signal.

3. A COMPLEXITY MEASURE OF DYADIC TREES

Consider a function $f : [0,1] \mapsto \mathbb{R}$ with maximum resolution 2^m. By this we mean that f is constant on the intervals $[2^{-m}l, 2^{-m}(l+1)), l = 0, 1, \ldots, 2^m - 1$. We can represent f using the sequence $\{\Psi_m(l)\}_{l=0}^{2^m-1}$ (with $\Psi_m(l) = f(2^{-m}l)$). One way to analyze the complexity of f is to compute the morphological wavelet coefficients of $\{\Psi_m(l)\}_{l=0}^{2^m-1}$, as suggested in the last section. In this section, we propose another way to analyze the complexity of f. We will reveal a connection between these two approaches in the next section.

Let γ be an arbitrary threshold and $S_\gamma = \{x \in [0,1] : f(x) \geq \gamma\}$ be the "level set" obtained by thresholding f at γ. In [3] a complexity measure of the set S_γ is proposed for the purpose of regularization. The idea is to represent S_γ using a dyadic tree. The tree starts with a root node corresponding to the whole interval $[0,1]$. Each node has two children corresponding to the left and right half of the interval, until each leaf node of the tree corresponds to an interval that is entirely inside or outside the set S_γ. Denote this tree by T_γ and let $\pi(T_\gamma)$ be the set of leaf nodes of T_γ. The following complexity measure is used as a regularization term in [3]:

$$\Phi(T_\gamma) = \sum_{L \in \pi(T_\gamma)} \phi(|L|) \tag{7}$$

where $\phi(|L|)$ only depends on the size of the leaf node $|L|$.

In our setting, since f has maximum resolution 2^m, $|L|$ only takes the values 2^{-k}, $k = 0, 1, \ldots, m$. Let

$$\alpha_k = \phi(2^{-k}), \quad k = 0, 1, \ldots, m. \tag{8}$$

We assume $\alpha_0 = 0$ so that the complexity measure of $[0,1]$ is 0, and $2\alpha_{k+1} > \alpha_k$ ($k \geq 1$) so that *splitting a node* actually *increases* the complexity measure $\Phi(T_\gamma)$. In level set estimation, $\{\alpha_k\}_{k=1}^m$ is usually a decaying sequence (see [4] for an example). This makes $\Phi(T_\gamma)$ a penalty term that favors unbalanced dyadic trees. In representing a level set using a dyadic tree, unbalanced trees have an advantage in accurately representing the boundaries of the level set. Also, the faster $\{\alpha_k\}_{k=1}^m$ decays, the more preference $\Phi(T_\gamma)$ has for unbalanced trees (since deeper nodes are penalized less), and the greater the ability of the tree to capture fine edge detail.

4. A CONNECTION: THE MAIN THEOREM

Let f and $\{\Psi_m(l)\}_{l=0}^{2^m-1}$ be as defined in the beginning of §3. Consider the complexity loss $\Phi(T_\gamma)$ for different values of γ and the morphological Haar wavelet coefficients of the sequence $\{\Psi_m(l)\}_{l=0}^{2^m-1}$. The main theorem of this paper reveals the following connection between these quantities.

Theorem 1. *Let f, S_γ, T_γ, $\Phi(T_\gamma)$, $\{\Psi_m(l)\}_{l=0}^{2^m-1}$, $\{W_{k,l}^\vee\}$, $\{W_{k,l}^\wedge\}$, $\{\alpha_k\}_{k=0}^m$ be as defined in previous sections. Then:*

$$\int \Phi(T_\gamma) d\gamma = \sum_{k=0}^{m-1} \sum_{l=0}^{2^k-1} \alpha_{k+1} \left(|W_{k,l}^\vee| + |W_{k,l}^\wedge| \right). \tag{9}$$

Proof. Define the following subintervals of $[0,1]$:

$$I_{k,l} = \{x : 0 \leq 2^k x - l < 1\} \tag{10}$$

for $k = 0, 1, \ldots, m$ and $l = 0, 1, \ldots, 2^k - 1$. These intervals correspond to all possible leaf nodes when using a dyadic tree T_γ to represent a level set $S_\gamma = \{x \in [0,1] : f(x) \geq \gamma\}$ in $[0,1]$. Moreover, define the dynamic range of f on the intervals $I_{k,l}$ to be

$$d_{k,l} = \max_{x \in I_{k,l}} f(x) - \min_{x \in I_{k,l}} f(x) \tag{11}$$

for $k = 0, 1, \ldots, m$, and $l = 0, 1, \ldots, 2^k - 1$.

We claim that:

$$\int \Phi(T_\gamma) d\gamma = \sum_{k=0}^{m-1} \sum_{l=0}^{2^k-1} (2\alpha_{k+1} - \alpha_k) d_{k,l}. \tag{12}$$

To prove this we will use the following lemma.

Lemma 1. *Let $[\![P]\!]$ denote the indicator function of condition P. Then*

$$\Phi(T_\gamma) = \sum_{k=0}^{m-1} \sum_{l=0}^{2^k-1} (2\alpha_{k+1} - \alpha_k) [\![\min_{x \in I_{k,l}} f(x) < \gamma \leq \max_{x \in I_{k,l}} f(x)]\!]. \tag{13}$$

Proof. By definition and $\alpha_0 = 0$:

$$\begin{aligned} \Phi(T_\gamma) &= \sum_{k=0}^{m} \sum_{l=0}^{2^k-1} \alpha_k [\![I_{k,l} \text{ is a leaf node in } T_\gamma]\!] \\ &= \sum_{k=1}^{m} \sum_{l=0}^{2^k-1} \alpha_k [\![I_{k,l} \text{ is a leaf node in } T_\gamma]\!]. \end{aligned} \tag{14}$$

So the problem reduces to analyzing whether $I_{k,l}$ is a leaf node in T_γ or not. The sufficient and necessary conditions for $I_{k,l}$ ($k \geq 1$) to be a leaf node in the representation of S_γ are:

1. Interval $I_{k,l}$ is either entirely contained in S_γ or entirely contained in $[0,1] \backslash S_\gamma$, so that the dyadic tree does not have to split $I_{k,l}$ into smaller intervals. This is equivalent to the opposite of the condition: $\min_{x \in I_{k,l}} f(x) < \gamma \leq \max_{x \in I_{k,l}} f(x)$. **And:**
2. the condition in (1) *does not* hold for interval $I_{k-1, \lfloor \frac{l}{2} \rfloor}$, so that the dyadic tree has to divide all the parent nodes of $I_{k,l}$ to reach $I_{k,l}$. This is equivalent to the condition: $\min_{x \in I_{k-1, \lfloor \frac{l}{2} \rfloor}} f(x) < \gamma \leq \max_{x \in I_{k-1, \lfloor \frac{l}{2} \rfloor}} f(x)$.

Using the above arguments, we can represent whether $I_{k,l}$ is a leaf node or not using the following formula:

$$\begin{aligned} & [\![I_{k,l} \text{ is a leaf node in } T_\gamma]\!] \\ = & [\![\min_{x \in I_{k-1, \lfloor \frac{l}{2} \rfloor}} f(x) < \gamma \leq \max_{x \in I_{k-1, \lfloor \frac{l}{2} \rfloor}} f(x)]\!] \\ & - [\![\min_{x \in I_{k,l}} f(x) < \gamma \leq \max_{x \in I_{k,l}} f(x)]\!]. \end{aligned} \tag{15}$$

Plugging this into (14) we find:

$$\Phi(T_\gamma) = \sum_{k=1}^{m}\sum_{l=0}^{2^k-1} \alpha_k [\![\min_{x\in I_{k-1,\lfloor \frac{l}{2}\rfloor}} f(x) < \gamma \leq \max_{x\in I_{k-1,\lfloor \frac{l}{2}\rfloor}} f(x)]\!]$$
$$- \sum_{k=1}^{m}\sum_{l=0}^{2^k-1} \alpha_k [\![\min_{x\in I_{k,l}} f(x) < \gamma \leq \max_{x\in I_{k,l}} f(x)]\!]$$
$$= \sum_{k=0}^{m-1}\sum_{l=0}^{2^{k+1}-1} \alpha_{k+1} [\![\min_{x\in I_{k,\lfloor \frac{l}{2}\rfloor}} f(x) < \gamma \leq \max_{x\in I_{k,\lfloor \frac{l}{2}\rfloor}} f(x)]\!]$$
$$- \sum_{k=1}^{m}\sum_{l=0}^{2^k-1} \alpha_k [\![\min_{x\in I_{k,l}} f(x) < \gamma \leq \max_{x\in I_{k,l}} f(x)]\!]$$
$$= \sum_{k=0}^{m-1}\sum_{l=0}^{2^{k+1}-1} \alpha_{k+1} [\![\min_{x\in I_{k,\lfloor \frac{l}{2}\rfloor}} f(x) < \gamma \leq \max_{x\in I_{k,\lfloor \frac{l}{2}\rfloor}} f(x)]\!]$$
$$- \sum_{k=0}^{m-1}\sum_{l=0}^{2^k-1} \alpha_k [\![\min_{x\in I_{k,l}} f(x) < \gamma \leq \max_{x\in I_{k,l}} f(x)]\!]$$
$$(\because \alpha_0 = 0 \text{ and } f \text{ is constant on } I_{m,l})$$
$$= 2\sum_{k=0}^{m-1}\sum_{l=0}^{2^k-1} \alpha_{k+1} [\![\min_{x\in I_{k,l}} f(x) < \gamma \leq \max_{x\in I_{k,l}} f(x)]\!]$$
$$- \sum_{k=0}^{m-1}\sum_{l=0}^{2^k-1} \alpha_k [\![\min_{x\in I_{k,l}} f(x) < \gamma \leq \max_{x\in I_{k,l}} f(x)]\!]$$
$$= \sum_{k=0}^{m-1}\sum_{l=0}^{2^k-1} (2\alpha_{k+1} - \alpha_k) [\![\min_{x\in I_{k,l}} f(x) < \gamma \leq \max_{x\in I_{k,l}} f(x)]\!].$$

□

Equation (12) now follows by integrating (13) over γ and using the following fact:

$$\int [\![\min_{x\in I_{k,l}} f(x) < \gamma \leq \max_{x\in I_{k,l}} f(x)]\!] d\gamma = d_{k,l}. \quad (16)$$

We now connect $d_{k,l}$ to the morphological Haar wavelet coefficients $W_{k,l}^{\vee}$ and $W_{k,l}^{\wedge}$. From the definition of morphological Haar wavelet transform (3)-(6), we have $\Psi_k^{\vee}(l) = \max_{x\in I_{k,l}} f(x)$ and $\Psi_k^{\wedge}(l) = \min_{x\in I_{k,l}} f(x)$, therefore:

$$d_{k,l} = \Psi_k^{\vee}(l) - \Psi_k^{\wedge}(l)$$
$$= \Psi_{k+1}^{\vee}(2l) \vee \Psi_{k+1}^{\vee}(2l+1) - \Psi_{k+1}^{\wedge}(2l) \wedge \Psi_{k+1}^{\wedge}(2l+1)$$
$$= \frac{1}{2}(\Psi_{k+1}^{\vee}(2l) + \Psi_{k+1}^{\vee}(2l+1) + |\Psi_{k+1}^{\vee}(2l) - \Psi_{k+1}^{\vee}(2l+1)|)$$
$$- \frac{1}{2}(\Psi_{k+1}^{\wedge}(2l) + \Psi_{k+1}^{\wedge}(2l+1) - |\Psi_{k+1}^{\wedge}(2l) - \Psi_{k+1}^{\wedge}(2l+1)|)$$
$$= \frac{1}{2}d_{k+1,2l} + \frac{1}{2}d_{k+1,2l+1} + \frac{1}{2}\left(|W_{k,l}^{\vee}| + |W_{k,l}^{\wedge}|\right).$$

So

$$2d_{k,l} - d_{k+1,2l} - d_{k+1,2l+1} = |W_{k,l}^{\vee}| + |W_{k,l}^{\wedge}|. \quad (17)$$

Now by plugging (17) into (12), we have:

$$\int \Phi(T_\gamma) d\gamma = \sum_{k=0}^{m-1}\sum_{l=0}^{2^k-1} (2\alpha_{k+1} - \alpha_k) d_{k,l}$$
$$= \sum_{k=0}^{m-1}\sum_{l=0}^{2^k-1} 2\alpha_{k+1} d_{k,l} - \sum_{k=0}^{m-1}\sum_{l=0}^{2^k-1} \alpha_k d_{k,l}$$
$$= \sum_{k=0}^{m-1}\sum_{l=0}^{2^k-1} 2\alpha_{k+1} d_{k,l} - \sum_{k=0}^{m-1}\sum_{l=0}^{2^{k+1}-1} \alpha_{k+1} d_{k+1,l} \quad (18)$$
$$(\because \alpha_0 = 0 \text{ and } d_{m,l} = 0)$$
$$= \sum_{k=0}^{m-1}\sum_{l=0}^{2^k-1} \alpha_{k+1}(2d_{k,l} - d_{k+1,2l} - d_{k+1,2l+1})$$
$$= \sum_{k=0}^{m-1}\sum_{l=0}^{2^k-1} \alpha_{k+1}\left(|W_{k,l}^{\vee}| + |W_{k,l}^{\wedge}|\right).$$

This proves the main theorem. □

5. APPLICATIONS OF THE MAIN THEOREM

The main theorem established an *equivalency* between a weighted L1 norm of the morphological Haar wavelet coefficients and the integral of a level set complexity measure. A special case is when f is an indicator function of level set $\tilde{S}$. By the following corollary, the regularization on a level set $\tilde{S}$ proposed in [3] is equivalent to a weighted L1 regularization of the morphological wavelet coefficients of the indicator function of $\tilde{S}$.

Corollary 1. *Let $\tilde{S}$ be a subset of $[0,1]$ with indicator function $f(x) = [\![x \in \tilde{S}]\!]$ and dyadic tree representation $T_{\tilde{S}}$. Assume that f is constant on the intervals $[2^{-m}l, 2^{-m}(l+1)), 0 \leq l \leq 2^m - 1$. Let $\{\Psi_m(l)\}_{l=0}^{2^m-1}$, $\{W_{k,l}^{\vee}\}$, $\{W_{k,l}^{\wedge}\}$, $\{\alpha_k\}_{k=0}^{m}$ be as previously defined. Then:*

$$\Phi(T_{\tilde{S}}) = \textstyle\sum_{k=0}^{m-1}\sum_{l=0}^{2^k-1} \alpha_{k+1}\left(|W_{k,l}^{\vee}| + |W_{k,l}^{\wedge}|\right). \quad (19)$$

Proof. Since f is an indicator function and $\alpha_0 = 0$, we have $\Phi(T_\gamma) = 0$ for $\gamma \leq 0$ or $\gamma > 1$, and $\Phi(T_\gamma) = \Phi(T_{\tilde{S}})$ for $0 < \gamma \leq 1$. Therefore $\Phi(T_{\tilde{S}}) = \int_0^1 \Phi(T_\gamma)d\gamma$ and

$$\textstyle\int_0^1 \Phi(T_\gamma)d\gamma = \sum_{k=0}^{m-1}\sum_{l=0}^{2^k-1} \alpha_{k+1}\left(|W_{k,l}^{\vee}| + |W_{k,l}^{\wedge}|\right).$$

□

The theorem also helps us to understand why and how shrinking morphological wavelet coefficients will preserve edges. Consider minimizing the following loss function:

$$\sum_{k=0}^{m-1}\sum_{l=0}^{2^k-1}\left((\hat{W}_{k,l}^{\vee} - W_{k,l}^{\vee})^2 + (\hat{W}_{k,l}^{\wedge} - W_{k,l}^{\wedge})^2\right)$$
$$+\lambda \sum_{k=0}^{m-1}\sum_{l=0}^{2^k-1} \alpha_{k+1}\left(|\hat{W}_{k,l}^{\vee}| + |W_{k,l}^{\wedge}|\right). \quad (20)$$

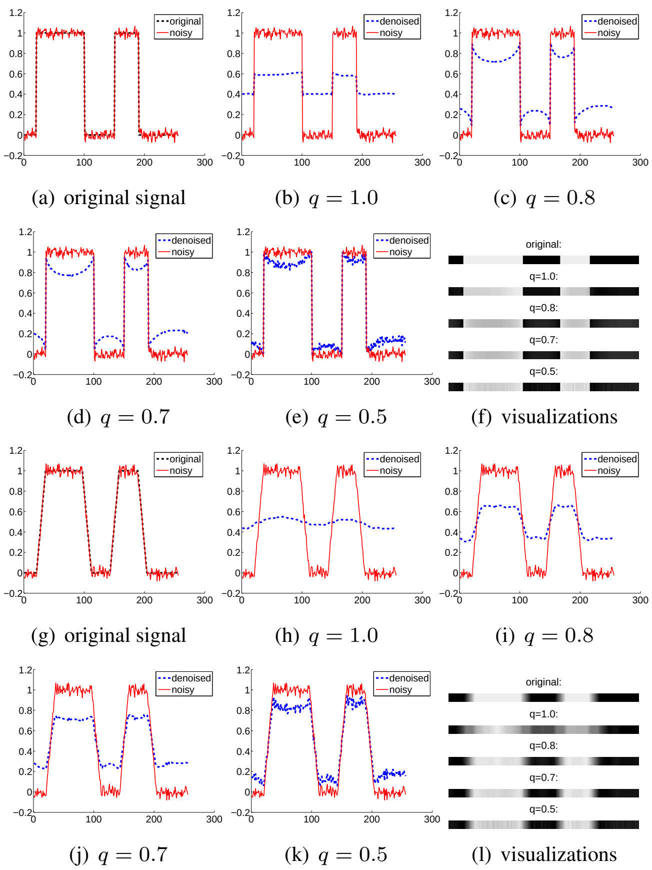

Fig. 1. Experiment with different q values to denoise signals with sharp edges (a-f) and smooth edges (g-l). ($\lambda = 1.6$)

The optimal solution for this problem is:

$$\hat{W}^{\vee}_{k,l} = g(W^{\vee}_{k,l}, \frac{\lambda\alpha_k}{2}), \quad \hat{W}^{\wedge}_{k,l} = g(W^{\wedge}_{k,l}, \frac{\lambda\alpha_k}{2}), \tag{21}$$

where $g(x,t) = sgn(x)(|x| - t)_+$ is the soft thresholding function. So soft thresholding morphological wavelet coefficients as in (21), is actually minimizing the loss function in (20), which is the sum of an L2 distance and a regularization term $\sum_{k=0}^{m-1}\sum_{l=0}^{2^k-1} \alpha_{k+1}\left(|\hat{W}^{\vee}_{k,l}| + |W^{\wedge}_{k,l}|\right)$. By the main theorem this latter term equals the integral of level set regularization terms. Therefore performing soft thresholding on morphological wavelet coefficients will minimize an L2 loss with the *same* regularization term used in level set estimation. This regularization term is known for preferring unbalanced tree structures well adapted for preserving edges.

In §3 we pointed out that a faster decay of $\{\alpha_k\}_{k=1}^m$ results in a level set regularization with more preference on unbalanced trees, therefore allowing more edge components to appear in the recovered signals. This naturally suggests the family of thresholding schemes with $\alpha_k = q^k$, where the parameter $q \leq 1$ can be used to control the edge sensitivity.

The examples in Fig.1 demonstrate the behavior of this thresholding scheme. We start with simple signals that have sharp or smooth edge features and add white Gaussian noise (Fig.1(a,g)). We then perform soft thresholding (21) with fixed λ and different q values. The denoised results are averaged over $\max$ and $\min$ versions of morphological wavelets and over all cycle spinnings ([5]). We also plot visualizations of the recovered signals by converting the amplitudes into image intensities. The signals are normalized so that the maximum value corresponds to white and the minimum value corresponds to black. This creates a fair comparison between recovered signals with different amounts of *shrinkage*. In Fig.1(l), while uniform thresholding ($q = 1$) results in blurred edges, smaller q values ($q = 0.8, 0.7$) exhibit greater edge sensitivity. The assumption $2\alpha_{k+1} > \alpha_k$ ($k \geq 1$) in §3 also dictates that $q > 1/2$. Otherwise the corresponding $\Phi(T_\gamma)$ does not even favor *simpler* trees. Therefore such thresholding scheme would not denoise signals at all (Fig.1(e,k)).

6. CONCLUSIONS

In this paper we established an interesting theoretical connection between morphological wavelets and regularized level set estimation based on the complexity of dyadic trees. This connection helps us to understand level set regularization proposed in [3] as a weighted L1 regularization on the coefficients of a nonlinear wavelet transform. We also connected a particular soft thresholding scheme for morphological wavelet coefficients to the unbalanced dyadic trees preferred in level set estimation. The simple illustrations in Fig.1 provide insight into why these nonlinear methods can be successful as well as when they will fail.

7. REFERENCES

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